Probability as a measure

Outline

1) What is a probability?
2) Measures and integrals
3) Densities
4) Probability spaces

What is probability?

Two common answers:

1) **Frequentist answer**: Relative frequency over many repetitions of a given experiment.

2) **Bayesian answer**: Degree of belief that something is true, or will happen.

Many disagreements, e.g. about what kinds of things we can meaningfully assign prob. to.

- Prob. (Die roll lands on 4)
- Prob. (Surgery is successful)
- Prob. (Harris wins upcoming election)
- Prob. (Subatomic particle has predicted mass)
- Prob. ($P = NP$)
- Prob. (20^{th} digit of $\sqrt{2}$ is 5)

Bayesians get mileage by putting probs on everything.

Frequentists try to avoid this.

Many controversies about this!

Mathematical Probability

Fortunately, these disagreements don't extend to the mathematical construct of probability

<u>Mathematical answer</u>: A function P mapping (some) subsets of a <u>sample space</u> $\mathcal{X}$ to $[0, 1]$, which is <u>additive</u> over disjoint sets:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \quad \text{if} \quad A_i \cap A_j = \emptyset, \text{ all } i \neq j$$

and has $P(\mathcal{X}) = 1$

Originally invented to analyze games of chance

Classical theory for discrete events defined by Laplace

Problems remained for continuous variables, e.g.

- treatment of pathological sets
- conditioning on probability zero events

Kolmogorov (1933) recognized:

- probability is a special case of a <u>measure</u>,
- expectation is an <u>integral</u> against a prob. measure

Measure theory basics

Measure theory is a rigorous grounding for probability theory [subject of 205A)

Simplifies notation & clarifies concepts, especially around integration & conditioning

Given a set $\mathcal{X}$, a measure μ maps subsets $A \subseteq \mathcal{X}$ to non-negative numbers $\mu(A) \in [0, \infty]$

Example $\mathcal{X}$ countable (e.g. $\mathcal{X} = \mathbb{Z}$)

Counting measure $\#(A) = \#$ points in A

Example $\mathcal{X} = \mathbb{R}^n$

Lebesgue measure
$$\lambda(A) = \int \cdots \int_A dx_1 \cdots dx_n = \text{Volume}(A)$$

Standard Gaussian distribution:

$$P_Z(A) = \mathbb{P}(Z \in A) \quad \text{where } Z \sim N(0, 1)$$
$$= \int_A \phi(x)\, dx \qquad \phi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

NB Because of pathological sets, $\lambda(A)$ can only be defined for certain subsets $A \subseteq \mathbb{R}^n$

In general, the domain of a measure μ is a collection of subsets $\mathcal{F} \subseteq 2^{\mathcal{X}}$ (power set)

$\mathcal{F}$ should satisfy certain closure properties (technical term: <u>σ-field</u>) (not important for us)

1. $\mathcal{X} \in \mathcal{F}$
2. If $A \in \mathcal{F}$ then $\mathcal{X} \setminus A \in \mathcal{F}$
3. If $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

<u>Ex</u>: $\mathcal{X}$ countable, $\mathcal{F} = 2^{\mathcal{X}}$

<u>Ex</u>: $\mathcal{X} = \mathbb{R}^n$, $\mathcal{F} =$ Borel σ-field $\mathcal{B}$

$\mathcal{B}$ = smallest σ-field including all open rectangles

$$(a_1, b_1) \times \cdots \times (a_n, b_n) \qquad a_i < b_i \quad \forall i$$

Given a <u>measurable space</u> $(\mathcal{X}, \mathcal{F})$ a <u>measure</u> is a map $\mu : \mathcal{F} \to [0, \infty]$ with

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \quad \text{for disjoint } A_1, A_2, \ldots \in \mathcal{F}$$

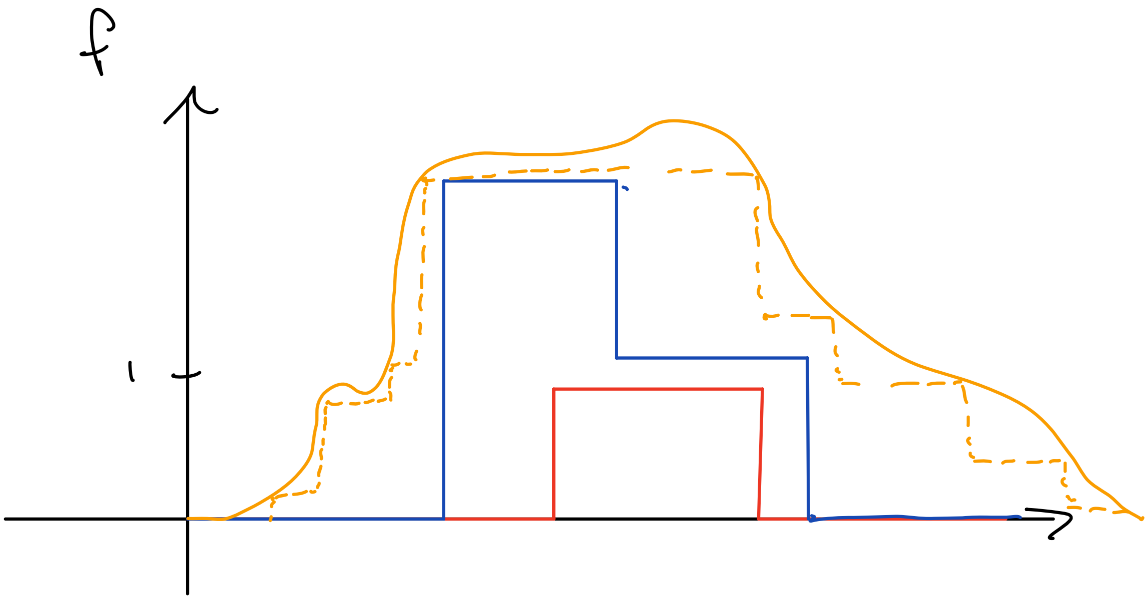
$$\mu(\emptyset) = 0$$

μ <u>probability measure</u> if $\mu(\mathcal{X}) = 1$

Integrals

Measures let us define <u>integrals</u> that put weight $\mu(A)$ on $A \subseteq \mathcal{X}$

Define $\int 1\{x \in A\}\, d\mu(x) = \mu(A)$, extend to other functions by linearity & limits:



Indicator $\int 1\{x \in A\}\, d\mu(x) = \mu(A)$

Simple Function $\int \left(\sum c_i 1\{x \in A_i\} \right) d\mu(x) = \sum c_i \mu(A_i)$

"Nice enough" (measurable) + function $\int f(x)\, d\mu(x)$ approximated by simple functions

Examples:

Counting: $\int f \, d\# = \sum_{x \in \mathcal{X}} f(x)$

Lebesgue integral

Lebesgue: $\int f \, d\lambda = \int \cdots \int f(x) \, dx_1 \cdots dx_n$

Gaussian: Note $\int 1_A(x) \, dP_Z(x) = P_Z(A) = \int_{-\infty}^{\infty} 1_A \, \phi \, dx$

By extension,

$$\int f \, dP_Z = \int f(x) \phi(x) \, dx = \mathbb{E}[f(Z)]$$

To evaluate $\int f \, dP_Z$ rewrite as $\int f \phi \, dx$.

density [can't always do this] e.g. Binom

It is nice to turn integrals we care about into Lebesgue integrals. When can we do this?

Densities

λ and P above are closely related. Want to make this precise.

Given $(\mathcal{X}, \mathcal{F})$, two measures P, μ

We say P is <u>absolutely continuous</u> wrt μ if $P(A) = 0$ whenever $\mu(A) = 0$

Notation: <u>$P \ll \mu$</u> or we say μ <u>dominates</u> P

If $P \ll \mu$ then (under mild conditions) we can always define a <u>density function</u>

$\rho : \mathcal{X} \to [0, \infty)$ with

$$P(A) = \int_A \rho(x) \, d\mu(x)$$

$$\int f(x) \, dP(x) = \int f(x) \rho(x) \, d\mu(x)$$

Sometimes written $\rho(x) = \frac{dP}{d\mu}(x)$, called <u>Radon-Nikodym derivative</u>

Densities are very useful:

Turn $\int f(x)\,dP(x)$ into something we know how to evaluate, such as

1) $\int_{\mathcal{X}} f(x)p(x)\,dx$ (X continuous, $\mathcal{X} \subseteq \mathbb{R}^n$)

$p(x)$ called <u>probability density function</u> (pdf)

2) $\sum_{x \in \mathcal{X}} f(x)\,p(x)$ (X discrete, $\mathcal{X}$ countable)

$p(x)$ called <u>probability mass function</u> (pmf)

Often define distributions by giving their density wrt some known measure, e.g.

<u>Ex</u>: Binom(n, θ) pmf: $p(x) = \theta^x (1-\theta)^{n-x} \binom{n}{x}$, $x = 0, \dots, n$

(density p wrt counting measure on $\mathcal{X} = \{0, \dots, n\}$)

Note this dist. has <u>no</u> density wrt Lebesgue:

$\int_{\{0,\dots,n\}} p(x)\,dx = 0$ for any function p

Probability space, random variables

Problem setup may have many random outcomes with complex relationships to one another

Convenient to start with abstract outcome $\omega \in \Omega$

- represents "everything that happens"
- quantities of interest are functions of ω

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a <u>probability space</u>

$\omega \in \Omega$ called <u>outcome</u>

$A \in \mathcal{F}$ called <u>event</u>

$\mathbb{P}(A)$ called <u>probability of A</u>

A <u>random variable</u> is a function $X : \Omega \to \mathcal{X}$

We say X has distribution Q $(X \sim Q)$ if

$$\mathbb{P}(X \in B) = \mathbb{P}(\{\omega : X(\omega) \in B\}) = Q(B)$$

$Q(B)$ is the <u>push-forward</u> of $\mathbb{P}$: $Q(B) = \mathbb{P} \circ X^{-1}(B)$

Idea applies more generally: if μ measure on $\mathcal{X}$,

$f : \mathcal{X} \to \mathcal{Y}$ induces new measure $\nu(B) = \mu(f^{-1}(B))$

Can write events involving many R.V.s:

$$\mathbb{P}(X > Y > Z \geq 0) = \mathbb{P}(\{\omega : \cdots\})$$

The <u>expectation</u> is an integral w.r.t. $\mathbb{P}$

$$\mathbb{E}[f(X,Y)] = \int_\Omega f(X(\omega), Y(\omega))\, d\mathbb{P}(\omega)$$

To do real calculations we must eventually boil $\mathbb{P}$ or $\mathbb{E}$ down to concrete integrals/sums/etc.

If $\mathbb{P}(A) = 1$ we say A occurs <u>almost surely</u>

More in Keener ch. 1, much more in Stat 205A