# Completeness

## Outline

1) Completeness
2) Ancillarity
3) Basu's Theorem

# Completeness

**Def** $T(X)$ is <u>complete</u> for $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$

if $\quad \mathbb{E}_\theta f(T(X)) = 0 \quad \forall\theta$

$\Rightarrow \quad f(T) \overset{a.s.}{=} 0 \quad \forall\theta$

[Name comes from a prior notion that

$\mathcal{P}^T = \{P_\theta^T : \theta \in \Theta\}$ is "complete basis"

wrt inner product $\langle f, P_\theta^T \rangle = \int f(t)\, dP_\theta^T(t)$ ]

(see HW 3)

<u>Ex.</u> (Cont'd) Laplace location family has minimal suff stat. $S = (X_{(i)})_{i=1}^n$. Complete?

No: Let $M(S) = \text{median}(X)$

$\bar{X}(S) = \frac{1}{n} \sum X_i$

$\mathbb{E}_\theta \bar{X} = \mathbb{E}_\theta M = \theta$ (by symmetry)

$\mathbb{E}_\theta [\bar{X}(S) - M(S)] = 0 \quad \forall\theta$

$S(X)$ still has "a lot of extra fluff"

Ex $X_1, \dots, X_n \overset{iid}{\sim} U[0, \theta] \quad \theta \in (0, \infty)$

Can show $T(X) = X_{(n)}$ min. suff. Complete?

Find density of $T(X)$:

$$\mathbb{P}_\theta(T \le t) = \left(\frac{t}{\theta} \wedge 1\right)^n = \left(\frac{t}{\theta}\right)^n \wedge 1$$

$$\Rightarrow p_\theta(t) = \frac{d}{dt} \mathbb{P}_\theta(T \le t)$$

$$= n \frac{t^{n-1}}{\theta^n} 1\{t \le \theta\}$$

Suppose $0 = \mathbb{E}_\theta f(T) \quad \forall \theta > 0$

$$= \frac{n}{\theta^n} \int_0^\theta f(t) t^{n-1} dt \quad \forall \theta > 0$$

$$\Rightarrow \int_0^\theta f(t) t^{n-1} dt = 0 \quad \forall \theta > 0$$

$$\Rightarrow f(t) t^{n-1} = 0 \quad \text{a.e. } t > 0$$

**Def** Assume $\mathcal{P} = \{P_\eta : \eta \in \Xi\}$ has densities

$$p_\eta(x) = e^{\eta' T(x) - A(\eta)} h(x)$$

If $T(X)$ satisfies no linear constraint $\left(\nexists\ \beta \neq 0, \alpha : \beta' T(X) \overset{a.s.}{=} \alpha\right)$ and $\Xi$ contains an open set, we say $\mathcal{P}$ is <u>full-rank</u>

If $\mathcal{P}$ is not full-rank we say it is <u>curved</u>

[Note: If $T(x)$ satisfies linear constraint, then $\mathcal{P}$ might still be full-rank for a lower-dim. sufficient statistic]

Proof in Lehmann & Romano, Thm. 4.3.1

**Theorem** If $\mathcal{P}$ is full rank then $T(X)$ is complete sufficient

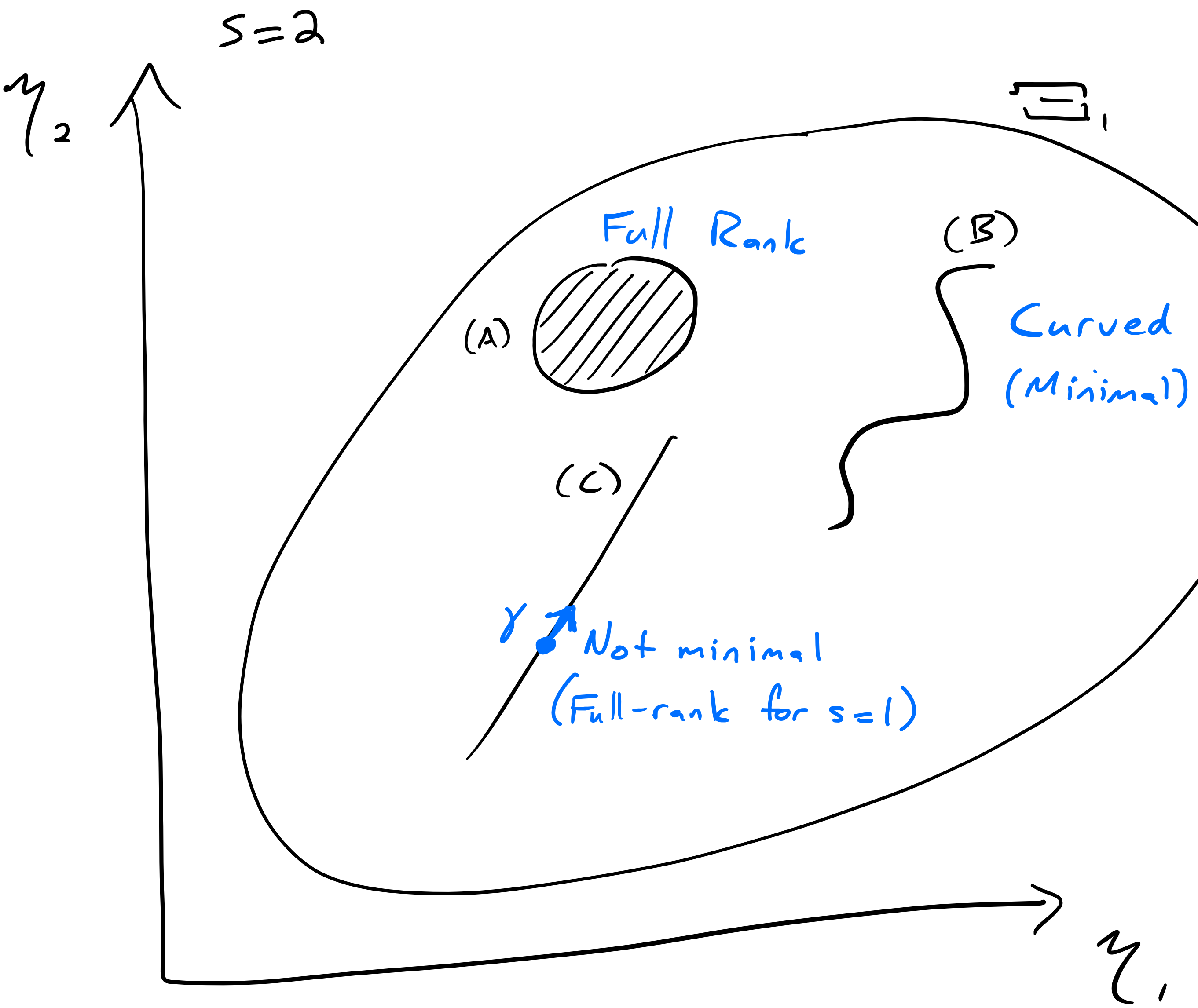
**Proof idea** wlog $T(X) = X$, $p_\eta(x) = e^{\eta' x - A(\eta)}$, $0 \in \overset{\circ}{\Xi}$ (interior)

Write $f(x) = f^+(x) - f^-(x)$, for $f^+, f^- \geq 0$

$$\underbrace{\int e^{\eta' x} f^+(x)\, d\mu(x)}_{\text{MGF for } Y^+ \sim \frac{f^+(x)}{\int f^+ d\mu}} = \underbrace{\int e^{\eta' x} f^-(x)\, d\mu(x)}_{\text{MGF for } Y^- \sim \frac{f^-(x)}{\int f^- d\mu}}$$

Uniqueness of MGFs $\Rightarrow Y^+ \overset{D}{=} Y^- \Rightarrow f^+ \overset{a.s.}{=} f^-$

# Diagram again




$T(X)$ definitely complete for (A)

Maybe not for (B), (C)

**Theorem** If $T(X)$ complete sufficient for $\mathcal{P}$ then $T(X)$ is minimal

Game plan for completeness proofs: show two things are a.s. equal by showing they have = expectation.

**Proof** Assume $S(X)$ is minimal suff

Let $\bar{T}(S(x)) = \mathbb{E}_{\cancel{\theta}}\left[T(X) \mid S(x)\right]$

← S suff.

Claim: $\bar{T}(S(x)) \overset{a.s.}{=} T(x)$

We have $S(X) \overset{a.s.}{=} f(T(x))$ (S minimal suff)

Let $g(t) = t - \bar{T}(f(t))$

$$\mathbb{E}_\theta\left[g(T(x))\right] = \mathbb{E}_\theta T(x) - \mathbb{E}_\theta\left[\bar{T}(S(x))\right]$$

$$= \mathbb{E}_\theta T(x) - \mathbb{E}_\theta\left[\mathbb{E}\left[T \mid S\right]\right]$$

$$= 0$$

$$\Rightarrow g(T(x)) \overset{a.s.}{=} 0 \quad \text{(completeness)}$$

$\boxtimes$

# Ancillarity

Two reasons to care about completeness:

1) Uniqueness of unbiased estimators using $T$

If $\mathbb{E}_\theta \delta_1(T) = \mathbb{E}_\theta \delta_2(T) = g(\theta), \forall \theta \in \Theta$

Then $\mathbb{E}_\theta[\delta_1 - \delta_2] = 0 \Rightarrow \delta_1 \overset{a.s.}{=} \delta_2$

[We will explore this further next time]

2) Basu's theorem: neat way to show independence

<u>Def</u> $V(X)$ is <u>ancillary</u> for $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ if its distribution does not depend on $\theta$. ($V$ carries no info. about $\theta$)

## (Aside:) Conditionality Principle

If $V(X)$ is ancillary then all inference should be conditional on $V(X)$

[will return to this in testing & CI unit

# Basu's Theorem

**Theorem** (Basu)

If $T(X)$ is complete sufficient and $V(X)$ is ancillary for $\mathcal{P}$, then

$$V(X) \perp\!\!\!\perp T(X) \quad \text{for all } \theta \in \Theta$$

**Proof**

Want $\mathbb{P}_\theta(V \in A, T \in B) = \mathbb{P}_\theta(V \in A)\,\mathbb{P}_\theta(T \in B)$ all $A, B, \theta$

Let
$$q_A(T(X)) = \mathbb{P}_{\cancel{\theta}}(V \in A \mid T)$$
T suff.

$$p_A = \mathbb{P}_{\cancel{\theta}}(V \in A)$$
V ancillary

$$\mathbb{E}_\theta\left[q_A(T) - p_A\right] = p_A - p_A = 0, \quad \forall \theta$$

$$\Rightarrow \quad q_A(T) \overset{a.s.}{=} p_A \quad \forall \theta$$

$$\mathbb{P}_\theta(V \in A, T \in B) = \int q_A(t)\, 1\{t \in B\}\, dP_\theta^T(t)$$
$$= p_A \int 1\{t \in B\}\, dP_\theta^T(t)$$
$$= \mathbb{P}(V \in A)\,\mathbb{P}_\theta(T \in B) \quad \boxtimes$$

# Using Basu's Theorem

Ancillarity, Completeness, Sufficiency are all properties wrt a family $\mathcal{P}$

Independence is a property of a distribution

If you can't verify the thm's hypotheses for one family, try a different family!

**Ex.** $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ $\mu \in \mathbb{R}, \sigma^2 > 0$

Sample mean $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

Sample variance $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

Want to show $\bar{X} \perp\!\!\!\perp S^2$

But neither stat. is ancillary or sufficient in the full family with $\mu, \sigma^2$ unknown

To apply Basu, use family with $\sigma^2$ known:

$$\mathcal{P} = \{ N(\mu, \sigma^2)^n : \mu \in \mathbb{R} \}$$

In $\mathcal{P}$, $\bar{X}$ is complete sufficient

and $S^2$ is ancillary since

$$S^2 = \sum (z_i - \bar{z})^2 \quad \text{for } z_i = X_i - \mu \overset{iid}{\sim} N(0, \sigma^2)$$

not statistics
but doesn't matter

Therefore $\bar{X} \perp\!\!\!\perp S^2$

[ Conclusion has nothing to do with "known"
or "unknown" parameters ]