Testing with one real parameter

Outline

1) One-sided tests in general
2) Two-sided tests
3) UMP unbiased tests

One-sided tests in general

$\mathcal{P} = \{P_\theta : \theta \in \Theta \subseteq \mathbb{R}\}$, $\quad \theta_0 \in \Theta$

$H_0 : \theta \overset{\geq}{\leq} \theta_0$ vs $H_1 : \theta \overset{<}{>} \theta_0$ called <u>one-sided hypothesis</u>

Often, no UMP test exists:

LRT may vary for different θ_1 values

If n large, could prioritize $\theta_1 = \theta_0 + \varepsilon$, $\varepsilon \downarrow 0$

$$\log LR(x) = \log \frac{p_{\theta_0+\varepsilon}(x)}{p_{\theta_0}(x)} \approx \varepsilon \cdot \dot{\ell}(\theta_0 ; x)$$

$\Rightarrow$ Use <u>score at θ_0</u> $\dot{\ell}(\theta_0 ; X)$ as test stat.

$$\phi(x) = 1\{\dot{\ell}(\theta_0 ; x) \geq c_\alpha\}$$

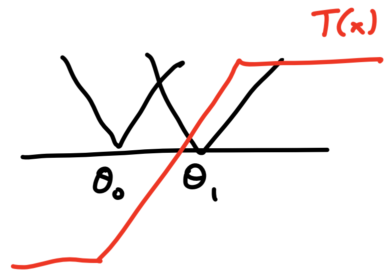
Need to check $\beta_\phi(\theta) \leq \alpha$ for $\theta \leq \theta_0$

Ex. Laplace: $X_1, \ldots, X_n \overset{iid}{\sim} \frac{1}{2} e^{-|x-\theta|}$

Test $H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$

$\theta_1 > \theta_0$: $$\log\left(p_{\theta_1}(x) / p_{\theta_0}(x) \right) = \sum_{i=1}^n |X_i - \theta_0| - |X_i - \theta_1|$$
$$= \sum T(X_i)$$

$$T(x) = \begin{cases} \theta_0 - \theta_1 & x \le \theta_0 \\ 2x - \theta_0 - \theta_1 & \theta_0 \le x \le \theta_1 \\ \theta_1 - \theta_0 & x \ge \theta_1 \end{cases}$$



Score $$\dot{\ell}(\theta_0; X) = \frac{d}{d\theta} \sum_i -|X_i - \theta| \Big|_{\theta = \theta_0}$$
$$= \sum_{i=1}^n \text{sign}(X_i - \theta_0)$$

Equivalent: $S(X) = \sum_{i=1}^n 1\{X_i \ge \theta_0\}$ **Sign test**

$$\sim \text{Binom}(n, \mathbb{P}_\theta(X_i > \theta_0))$$
$$\overset{\theta = \theta_0}{=} \text{Binom}(n, 1/2)$$

Nonparametric example $X_i \overset{iid}{\sim} F$ (cdf), $\theta(F) = \text{median}(F)$

Test $H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$

$S(X) \sim \text{Binom}(n, 1 - F(\theta_0)) = \text{Binom}(n, 1/2)$ if $\theta(F) = \theta_0$

Stochastically incr.

Def A real-valued statistic $T(X)$ is <u>stochastically increasing</u> in θ if

$\mathbb{P}_\theta(T(X) \le t)$ is non-incr. in θ, $\forall t$

If $\phi(x)$ rejects for large $T(x)$:

$$\phi(x) = 1\{T(x) > c\} + \gamma 1\{T(x) = c\}$$

and $T(X)$ is stochastically increasing in θ,

$$\mathbb{E}_\theta \phi(X) = (1-\gamma)\mathbb{P}_\theta(T > c) + \gamma \mathbb{P}_\theta(T \ge c) \nearrow \text{in } \theta$$

Ex $X_i \overset{iid}{\sim} p(x-\theta)$ (<u>location family</u>)

$T(X)$ = sample mean, median, sign statistic

Ex $X_i \overset{iid}{\sim} \frac{1}{\theta} p(x/\theta)$ (<u>scale family</u>)

$T(x) = \sum X_i^2$ or $\text{median}(|X_1|, \dots, |X_n|)$

Two-sided Alternatives

Setup: $\mathcal{P} = \{ P_\theta : \theta \in \Theta \subseteq \mathbb{R} \}$, $\theta_0 \in \Theta^\circ$

Test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$

(Can be generalized naturally to $H_0 : \theta \in [\theta_1, \theta_2]$)

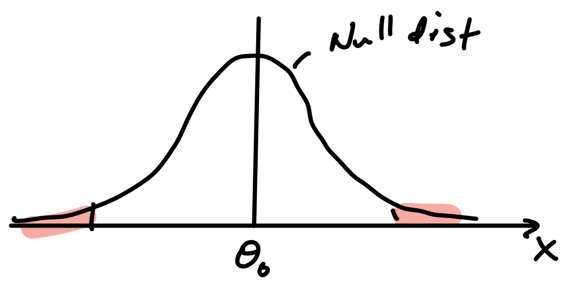
Two-tailed test rejects when $T(X)$ is "extreme"

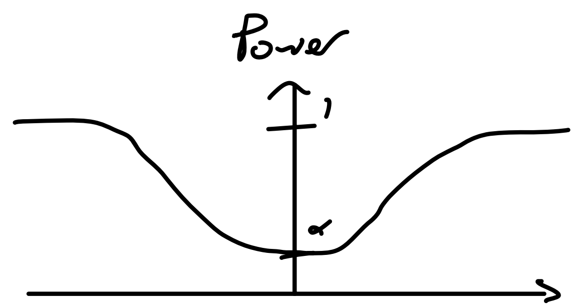
$$\phi(x) = \begin{cases} 1 & T(X) > c_2 \text{ or } T(X) < c_1 \\ 0 & T(X) \in (c_1, c_2) \\ \gamma_i & T(X) = c_i \end{cases}$$

Two ways to reject. How to balance?

For symmetric distributions like $N(\theta, 1)$,
natural choice is to equalize "lobes" of rej. region

$$\phi_2(x) = 1\{ |X - \theta_0| > z_{\alpha/2} \} \text{ for } H_0 : \theta = \theta_0$$





For asymmetric dists, or interval null $H_0 : \theta \in [\theta_1, \theta_2]$,
more complicated

Equal-tailed & unbiased tests

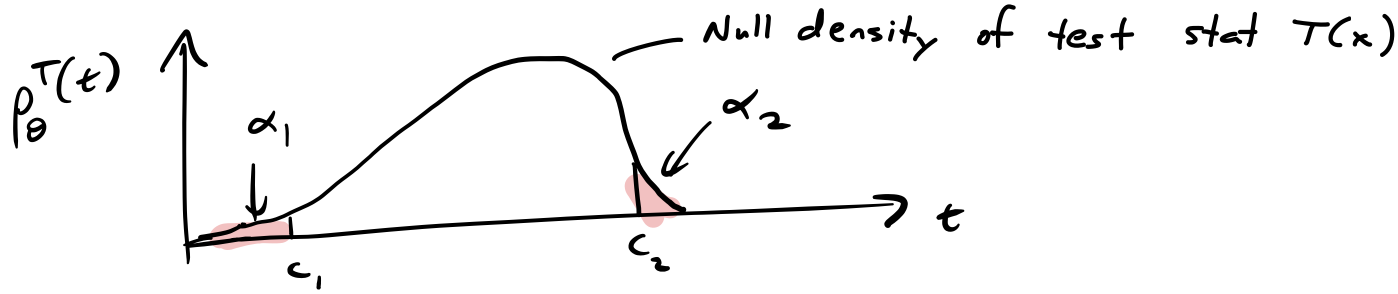
Point null ($H_0 : \theta = \theta_0$)

Let $\alpha_1 = \mathbb{P}_{\theta_0}(T < c_1) + \gamma_1 \mathbb{P}_{\theta_0}(T = c_1)$

$\alpha_2 = \mathbb{P}_{\theta_0}(T > c_2) + \gamma_2 \mathbb{P}_{\theta_0}(T = c_2)$

Valid if $\alpha_1 + \alpha_2 = \alpha$ (α_1 is "free parameter")

Idea 1: Equal-tailed test: $\alpha_1 = \alpha_2 = \frac{\alpha}{2}$



Ex $X \sim \text{Exp}(\theta)$, test $H_0 : \theta = 1$

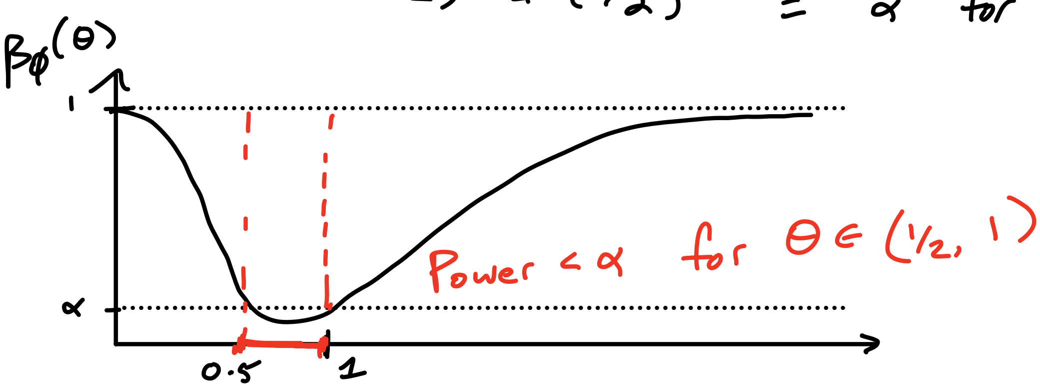
Solve for cutoffs: $\frac{\alpha}{2} = \mathbb{P}_1(X \le c_1) = 1 - e^{-c_1} \Rightarrow c_1 = -\log(1 - \alpha/2)$

$1 - \frac{\alpha}{2} = 1 - e^{-c_2} \Rightarrow c_2 = -\log(\alpha/2)$

$$\phi(x) = 1\{X < -\log(1-\alpha/2)\} + 1\{X > -\log(\alpha/2)\}$$

$$\beta_\phi(\theta) = \mathbb{P}_\theta\left\{\overset{\text{Exp}(1)}{\frac{X}{\theta}} < \frac{-\log(1-\alpha/2)}{\theta}\right\} + \mathbb{P}_\theta\left\{\frac{X}{\theta} > \frac{-\log(\alpha/2)}{\theta}\right\}$$

$$= 1 - (1-\alpha/2)^{1/\theta} + (\alpha/2)^{1/\theta} = \alpha \text{ for } \theta = 1 \text{ or } 1/2$$



Unbiased tests

Def $\phi(X)$ is <u>unbiased</u> if $\inf_{\theta \in \Theta_1} \mathbb{E}_\theta \phi(X) \geq \alpha$

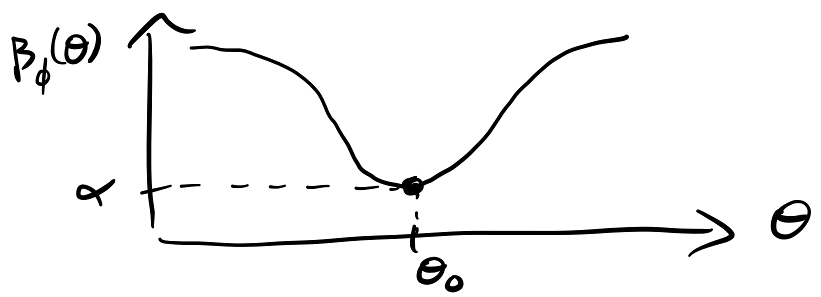
Idea 2: <u>Unbiased test</u>: ensure $\min \beta_\phi(\theta) = \alpha$

Choose c_1, γ_1 and c_2, γ_2 to solve:

$$\beta_\phi(\theta_0) = \alpha$$

$$\frac{d\beta_\phi}{d\theta}(\theta_0) = 0$$

(2 equations, "2" unknowns)



<u>Ex</u>: 1-parameter exp. family, $H_0: \eta = \eta_0$ vs $H_1: \eta \neq \eta_0$

$$X \sim e^{\eta T(x) - A(\eta)} h(x) \qquad (\text{MLR in } T(X))$$

Assume $T(X)$ continuous, solve

$$\alpha = \beta_\phi(\eta_0) = \mathbb{P}_{\eta_0}(T < c_1) + \mathbb{P}_{\eta_0}(T > c_2)$$

$$0 = \frac{d\beta_\phi}{d\eta}(\eta_0) = \text{Cov}_{\eta_0}(\phi(T), T)$$

$$= \mathbb{E}_{\eta_0}\left[(\phi(T) - \alpha)\, T(X)\right]$$

Theorem Assume $X \sim e^{\theta T(x) - A(\eta)} h(x)$

$H_0: |\theta - \theta_0| \le \delta$ vs $H_1: |\theta - \theta_0| > \delta$, $\delta \ge 0$

Let ϕ^* be test that rejects for extreme $T(X)$, with $c_1, c_2, \gamma_1, \gamma_2$ chosen so:

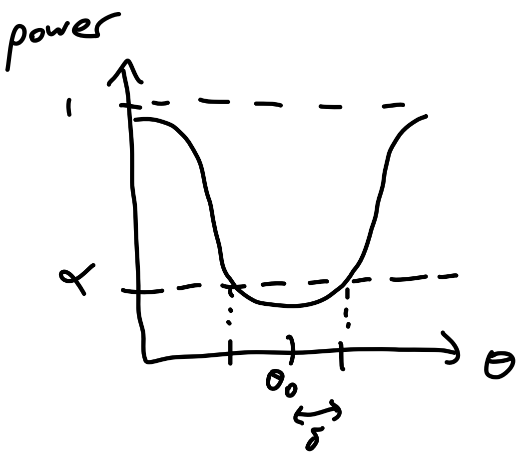
(i) $\beta_{\phi^*}(\theta_0 + \delta) = \beta_{\phi^*}(\theta_0 - \delta) = \alpha$

and, if $\delta = 0$ (point null)

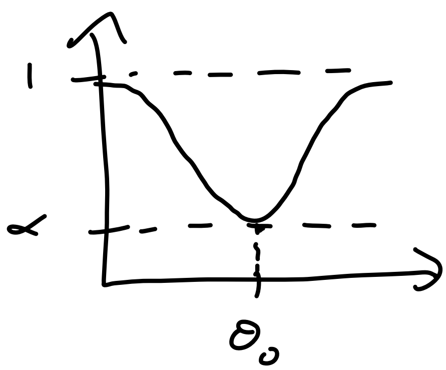
(ii) $0 = \dot{\beta}_{\phi_*}(\theta_0) = \mathbb{E}_{\theta_0}\left[(T - \mathbb{E}_{\theta_0} T)\, \phi(x)\right]$

Then ϕ^* is UMPU

$\delta > 0$:



$\delta = 0$:



Proof: Assume wlog $\theta_0 = 0$

$(\delta = 0)$:

Want to solve

$$\text{maximize} \quad \int \phi \, p_{\theta_1} \, d\mu$$

$$\text{s.t.} \quad \int \phi \, p_0 \, d\mu = \alpha \quad \text{(unbiased)}$$

$$\int \phi \, (T - \mathbb{E}_0 T) \, p_0 \, d\mu = 0$$

Lagrange form:

$$\max \int \phi \left(p_{\theta_1} - \lambda_1 p_0 - \lambda_2 p_0 (T - \mathbb{E}_0 T) \right) d\mu$$

$$= \int \phi \left(\frac{p_{\theta_1}}{p_0} - \lambda_1 - \lambda_2 (T - \mathbb{E}_0 T) \right) dP_0$$

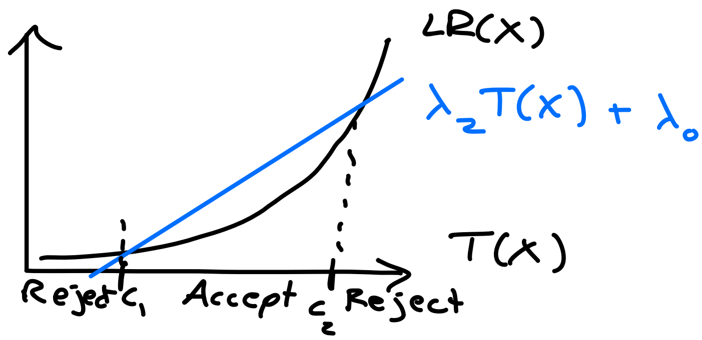
$$\Rightarrow \quad \phi^*(x) = \begin{cases} 1 & LR(x) > \lambda_2 T(x) + \lambda_0 \\ 0 & LR(x) < \lambda_2 T(x) + \lambda_0 \\ \text{arb.} & LR(x) = \lambda_2 T(x) + \lambda_0 \end{cases}$$

$\lambda_0 = \lambda_1 + \lambda_2 \mathbb{E}_0 T$

$$LR(X) = e^{\theta_1 T(x) + A(0) - A(\theta_1)}$$

$\theta_1 > 0$:

can find λ_1, λ_2 to match c_1, c_2



$\theta_1 < 0$ similar

Suppose ϕ, ϕ^* satisfy constraints,

ϕ^* maximizes Lagrangian for λ_1, λ_2

$$\beta_\phi(\theta_1) = \beta_\phi(\theta_1) + \lambda_1(\beta_\phi(0) - \alpha) + \lambda_2 \dot{\beta}_\phi(0)$$

$$\leq \beta_{\phi^*}(\theta_1) + \lambda_1(\beta_{\phi^*}(0) - \alpha) + \lambda_2 \dot{\beta}_{\phi^*}(0)$$

$$= \beta_{\phi^*}(\theta_1)$$

$(\delta > 0)$ $\max \int \phi p_{\theta_1} d\mu$

s.t. $\int \phi p_\delta d\mu = \int \phi p_{-\delta} d\mu = \alpha$

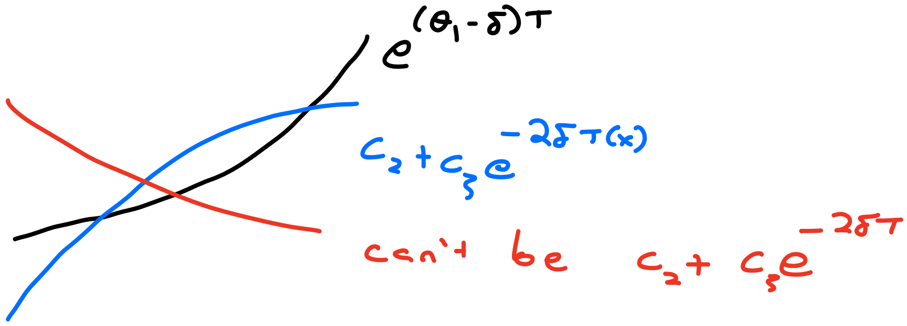
Lagrangian:

$$\int \phi(p_{\theta_1} - \lambda_1 p_\delta - \lambda_2 p_{-\delta}) d\mu$$

$$= \int \phi(c_1 e^{\theta_1 T} - c_2 e^{\delta T} - c_3 e^{-\delta T}) dP_0 \qquad c_1 > 0$$

$\theta_1 > \delta$:

Reject for $c_1 e^{(\theta_1 - \delta)T(x)} > c_2 + c_3 e^{-2\delta T(x)}$



Rest of proof same as $\delta = 0$