Outline

1) Wald test
2) Score test
3) Generalized likelihood ratio test
4) Asymptotic Relative Efficiency

Likelihood-Based Inference

Setting $X_1, \dots, X_n \overset{iid}{\sim} p_\theta(x)$, $p_\theta(x)$ "smooth" in θ

Assume $\mathbb{E}_\theta \nabla \ell_1(\theta; X_i) = 0$,

$$\text{Var}_\theta\left[\nabla \ell_1(\theta; X_i)\right] = -\mathbb{E}_\theta \nabla^2 \ell_1(\theta; X_i) = J_1(\theta) \succ 0,$$

$$\hat{\theta}_{MLE} \xrightarrow{P_\theta} \theta \quad \text{(consistent)}$$

Then, if $\theta = \theta_0$:

$$\frac{1}{\sqrt{n}} \nabla \ell_n(\theta_0; X) \Rightarrow N_d(0, J_1(\theta_0))$$

$$\frac{1}{n} \nabla^2 J_n(\theta_0; X) \xrightarrow{P} J_1(\theta_0)$$

Used $0 = \nabla \ell_n(\hat{\theta}_n) \approx \nabla \ell_n(\theta_0) + \nabla^2 \ell_n(\theta_0)(\hat{\theta}_n - \theta_0)$

to get $\sqrt{n}(\hat{\theta}_n - \theta_0) \Rightarrow N_d(0, J_1(\theta_0)^{-1})$

Can use this for inference on θ_0!

Wald-Type Confidence Regions

Assume we have some estimator $\hat{J}_n \succ 0$ s.t. $\frac{1}{n}\hat{J}_n \xrightarrow{P} J_1(\theta_0) \succ 0$. Then we can plug in:

If $\sqrt{n}\,(\hat{\theta}_n - \theta_0) \Rightarrow N_d(0, J_1(\theta_0)^{-1})$

then $(J_1(\theta_0))^{1/2}\sqrt{n}(\hat{\theta}_n - \theta_0) \Rightarrow N_d(0, I_d)$

so $\hat{J}_n^{1/2}(\hat{\theta}_n - \theta_0) \Rightarrow N_d(0, I_d)$ (Slutsky)

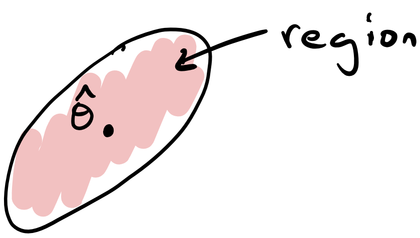
Leads to test of $H_0: \theta = \theta_0$:

$$\|\hat{J}_n^{1/2}(\hat{\theta}_n - \theta_0)\|^2 \Rightarrow \chi^2_d \quad \text{(Reject if large)}$$

So, $\mathbb{P}_{\theta_0}\left(\hat{J}_n^{1/2}(\hat{\theta}_n - \theta_0) \geq \underbrace{\chi^2_d(\alpha)}_{1-\alpha \text{ quantile}}\right) \to \alpha$

Note we reject θ_0 iff $\|\hat{J}_n^{1/2}(\hat{\theta}_n - \theta_0)\|^2 > \chi^2_d(\alpha)$

$\Longleftrightarrow$ reject θ_0 iff $\theta_0 \notin \underbrace{\hat{\theta}_n + \hat{J}_n^{-1/2} B_{\chi_d(\alpha)}(0)}_{\text{confidence ellipsoid}}$



More info $\Leftrightarrow$ smaller ellipse (shrinks like $1/\sqrt{n}$)

Options for $\hat{J}_n$:

1) Most obvious is to "plug in" the MLE:

(MLE for $J_n(\theta)$)

$$\hat{J}_n = J_n(\hat{\theta}_n)$$

$$= \mathrm{Var}_\theta(\nabla \ell_n(\theta; X))\Big|_{\theta=\hat{\theta}_n}$$

$$\text{(NB)} \neq \mathrm{Var}_{\hat{\theta}_n}(\nabla \ell_n(\hat{\theta}_n(X); X)) = 0$$

Or, $$\hat{J}_n = -\mathbb{E}_\theta \nabla^2 \ell_n(\theta)\Big|_{\theta=\hat{\theta}_n}$$

2) Observed Fisher info:

$$\hat{J}_n = -\nabla^2 \ell_n(\hat{\theta}_n; X)$$

Remarks:

- Both have $\frac{1}{n}\hat{J}_n \xrightarrow{P} J_1(\theta_0)$ in "nice" iid sampling setting
- Both make sense outside of iid setting
- Heuristically, plug-in measures info about θ in "typical" data set but obs. info. measures info about θ in "this" data set

Wald interval for θ_j:

If $\hat{\theta}_n \approx N_d(\theta_0, J_n(\theta_0)^{-1})$

then $\hat{\theta}_{n,j} \approx N_d(\theta_{0,j}, \underbrace{(J_n(\theta_0)^{-1})_{jj}}_{s.e.(\hat{\theta}_{n,j})^2})$

Leads to univariate interval:

$$C_j = \hat{\theta}_{nj} \pm \widehat{s.e.}(\hat{\theta}_{n,j}) \cdot z_{\alpha/2}$$

$$= \hat{\theta}_{nj} \pm \sqrt{(\hat{J}_n^{-1})_{jj}} \cdot z_{\alpha/2}$$

glm function in R uses these intervals / p-values, with $\hat{J}_n = -\nabla^2 \ell(\hat{\theta}_n)$

Conf. ellipsoid for $\theta_{0,S} = (\theta_{0,j})_{j \in S}$: $(|S| = k)$

$$\hat{\theta}_{n,S} \approx N_k(\theta_{0,S}, (J_n(\theta_0)^{-1})_{SS})$$

$$\rightsquigarrow C_S = \hat{\theta}_{n,S} + ((\hat{J}_n^{-1})_{SS})^{1/2} B_{\chi_k(\alpha)}(0)$$

More generally, if $\sqrt{n}(\hat{\theta}_n - \theta_0) \Rightarrow N_d(0, \Sigma(\theta_0))$

and $\frac{1}{n}\hat{\Sigma}_n \xrightarrow{P_{\theta_0}} \Sigma(\theta_0)$ ($\hat{\theta}_n$ not nec. MLE)

then we can do the same things

Ex Generalized linear model with fixed x

$x_1, \dots, x_n \in \mathbb{R}^d$ fixed

$$Y_i \overset{ind.}{\sim} p_{\eta_i}(y) = e^{\eta_i y_i - A(\eta_i)} h(y_i)$$

$$\eta_i = \beta' x_i \quad (\underline{\text{canonical form}})$$

Let $\mu_i(\beta) = \mathbb{E}_\beta Y_i$ $\left(= \mu(\eta_i(\beta)) \right)$

(more general: $f(\mu_i) = \beta' x_i$ for $\underline{\text{link fcn}}$ f)

Most common examples:

Logistic regression: $Y_i \overset{ind.}{\sim} \text{Bern}\left(\frac{e^{x_i'\beta}}{1+e^{x_i'\beta}} \right)$

Poisson log-linear model: $Y_i \overset{ind.}{\sim} \text{Pois}(e^{x_i'\beta})$

$$\ell_n(\beta; Y) = \sum_i (x_i'\beta) y_i - A(x_i'\beta) - \log h(y_i)$$

$$\nabla \ell_n(\beta; Y) = \sum_i y_i x_i - \dot{A}(x_i'\beta) \cdot x_i$$

$$= \sum_i (y_i - \mu_i(\beta)) x_i$$

$$-\nabla^2 \ell_n(\beta; Y) = \sum_i \ddot{A}(x_i'\beta) \cdot x_i x_i'$$

$$= \sum_i \text{Var}_\beta(y_i) \cdot x_i x_i'$$

$$= \text{Var}_\beta(\nabla \ell_n(\beta; Y))$$

(Not random)

$$(-\nabla^2 \ell_n(\beta_0))^{-1/2} \nabla \ell_n(\beta_0) \sim (0, I_d) \quad \text{in finite samples}$$

$$\overset{*}{\Rightarrow} N_d(0, I_d)$$

* Under regularity cond. on $X = \begin{pmatrix} -x_1- \\ \vdots \\ -x_n- \end{pmatrix}$

Taylor expansion of ℓ_n leads to

$$\hat{J}_n^{1/2} (\hat{\beta}_n - \beta) \Rightarrow N_d(0, I_d)$$

Advantages of Wald test:

1) Easy to invert, simple conf. regions
2) Asymptotically correct

Disadvantages:

1) Have to compute MLE
2) Depends on parameterization
3) Relies on two approximations:
   $\nabla \ell_n \approx$ Normal and $\ell_n \approx$ quadratic
4) Need MLE to be consistent
5) Confidence interval/ellipsoid might go outside Θ!

Score Test

Test $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$

We can bypass quadratic approximation entirely by using score as test stat

$$\frac{1}{\sqrt{n}} \nabla \ell_n(\theta_0; X) \xRightarrow{P_{\theta_0}} N_d(0, J_1(\theta_0))$$

$$\left(\text{or} \quad J_n(\theta_0)^{-1/2} \nabla \ell_n(\theta_0; X) \xRightarrow{P_{\theta_0}} N_d(0, I_d) \right)$$

So, we can reject $H_0: \theta = \theta_0$ if

$$\| J_n(\theta_0)^{-1/2} \nabla \ell_n(\theta_0; X) \|_2^2 \geq \chi_d^2(\alpha)$$

$d=1$: $\quad \dot{\ell}_n(\theta_0) / \sqrt{J_n(\theta_0)} \Rightarrow N(0,1)$,

can do 1-sided tests

Remarks

- No quadratic approx., no MLE
- No need to estimate Fisher info at θ_0

Can be generalized to case with nuisance params
Typically estimate via MLE on Θ_0

Score test is invariant to reparameterization*:

Assume $d=1$, $\theta = g(\zeta)$, $\dot{g}(\zeta) > 0 \ \forall \zeta$

$$q_\zeta(x) = p_{g(\zeta)}(x)$$

$$\dot{\ell}^{(\zeta)}(\zeta; X) = \frac{d}{d\zeta} \log p_{g(\zeta)}(x)$$

$$= \dot{\ell}^{(\theta)}(g(\zeta); X) \cdot \dot{g}(\zeta)$$

$$J^{(\zeta)}(\zeta) = J^{(\theta)}(g(\zeta)) \cdot \dot{g}(\zeta)^2$$

So $\dot{\ell}^{(\zeta)}(\zeta_0; X) \Big/ \sqrt{J^{(\zeta)}(\zeta_0)} \overset{a.s.}{=} \dot{\ell}^{(\theta)}(\theta_0; X) \Big/ \sqrt{J^{(\theta)}(\theta_0)}$

if $\theta_0 = g(\zeta_0)$

Ex s-parameter exp. fam:

$$X_1, \ldots, X_n \overset{iid}{\sim} e^{\eta' T(x) - A(\eta)} h(x)$$

$$\nabla \ell_n(\eta; X) = \Sigma T(X_i) - n\mu(\eta)$$

$$\| J_n(\eta_0)^{-1/2} (\Sigma T(X_i) - n\mu(\eta_0)) \|^2 \Rightarrow \chi^2_d$$

$$\frac{\Sigma T(X_i) - n\mu(\eta_0)}{\sqrt{n \operatorname{Var}_{\eta_0}(T(X_i))}} \overset{P_{\eta_0}}{\Rightarrow} N(0, 1)$$

Ex Pearson's χ^2 test (goodness of fit)

$$N = (N_1, \dots, N_d) \sim \text{Multinom}(n, (\pi_1, \dots, \pi_d))$$

$$= \frac{n!\, \pi_1^{N_1} \cdots \pi_d^{N_d}}{N_1! \cdots N_d!} \, 1\{\Sigma N_i = n\}$$

Note $\Sigma \pi_j = 1$ so this is a full-rank $(d-1)$-parameter exp. family, e.g.

$$\pi_j = \begin{cases} \dfrac{1}{1 + \sum_{k>1} e^{\eta_k}} & j = 1 \\[2ex] \dfrac{e^{\eta_j}}{1 + \sum_{k>1} e^{\eta_k}} & j > 1 \end{cases}$$

$$\nabla \ell_n(\eta; N) = (N_2, \dots, N_d) - (n\pi_2, \dots, n\pi_d)$$

$$\text{Var}_\eta(\nabla \ell(\eta)) = \begin{pmatrix} n\pi_2(1-\pi_2) & & -n\pi_i\pi_j \\ & \ddots & \\ -n\pi_i\pi_j & & n\pi_d(1-\pi_d) \end{pmatrix}$$

$$= n\left(\text{diag}(\pi_{2:d}) - \pi_{2:d}\pi_{2:d}'\right)$$

$$\Rightarrow J_n(\eta)^{-1} = \frac{1}{n} \cdot \left(\text{diag}(\pi_{2:d})^{-1} - \pi_1^{-1} 1 1'\right)$$

(uses $(A + uv')^{-1} = A^{-1} - \dfrac{A^{-1}uv'A^{-1}}{1 + v'A^{-1}u}$)

Score test of $H_0: \pi = \pi_0$:

don't really need asy. approx

(algebra)

$$\nabla \ell_n(\eta_0) J_n^{-1}(\eta_0) \nabla \ell_n(\eta_0) = \sum_{j=1}^d \frac{(N_j - n\pi_{0j})^2}{n\pi_{0j}} \overset{P_{\pi_0}}{\Rightarrow} \chi^2_{d-1}$$

Generalized LRT

Test $\quad H_0: \theta = \theta_0 \quad$ vs. $\quad H_1: \theta \neq \theta_0$

Taylor expand around $\hat{\theta}_n$:

$$\ell_n(\theta_0) - \ell_n(\hat{\theta}_n) = \cancelto{0}{\nabla \ell(\hat{\theta}_n)} + \frac{1}{2}(\theta_0 - \hat{\theta}_n)' \nabla^2 \ell_n(\tilde{\theta}_n)(\theta_0 - \hat{\theta}_n)$$

$$= -\frac{1}{2} \cdot \Big\| \Big(\underbrace{-\frac{1}{n} \nabla^2 \ell_n(\tilde{\theta}_n)}_{\xrightarrow{P} J_1(\theta_0)} \Big)^{1/2} \underbrace{\left(\sqrt{n}\,(\theta_0 - \hat{\theta}_n)\right)}_{\Rightarrow N(0, J_1(\theta_0)^{-1})} \Big\|_2^2$$

$$\Rightarrow -\frac{1}{2} \chi^2_d$$

Test stat: $\quad 2\left(\ell_n(\hat{\theta}_n; X) - \ell_n(\theta_0; X)\right) \overset{P_{\theta_0}}{\Rightarrow} \chi^2_d$

Composite vs. Composite:

$H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta \setminus \Theta_0$,

Assume
- $\Theta = \mathbb{R}^d$, Θ_0 d_0-dim manifold
- $\theta_0 \in \mathrm{relint}(\Theta_0)$
- $\hat{\theta}_n \xrightarrow{P_{\theta_0}} \theta_0$
- Likelihood "smooth"

Then $2(\ell_n(\hat{\theta}_n) - \ell_n(\hat{\theta}_0)) \Rightarrow \chi^2_{d-d_0}$

where $\hat{\theta}_0 = \underset{\theta \in \Theta_0}{\operatorname{argmin}}\ \ell_n(\theta; X)$

Why? Assume wlog $\theta_0 = 0$, $J_1(0) = I_d$ (reparam.)

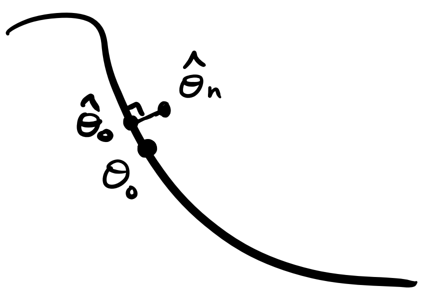
Then $\hat{\theta}_n \approx N_d(\theta_0, \frac{1}{n} I_d)$

And locally, $\nabla^2 \ell_n(\theta) \approx -n I_d$ near θ_0

$$\ell_n(\theta) - \ell_n(\hat{\theta}_n) \approx \frac{n}{2} \|\theta - \hat{\theta}_n\|^2$$

$$\hat{\theta}_0 \approx \underset{\theta \in \Theta_0}{\operatorname{argmin}} \|\theta - \hat{\theta}_n\| = \mathrm{Proj}_{\Theta_0}(\hat{\theta}_n)$$

$$2(\ell_n(\hat{\theta}_0) - \ell_n(\hat{\theta}_n)) \approx n \|\hat{\theta}_n - \mathrm{Proj}_{\Theta_0}(\hat{\theta}_n)\|^2$$



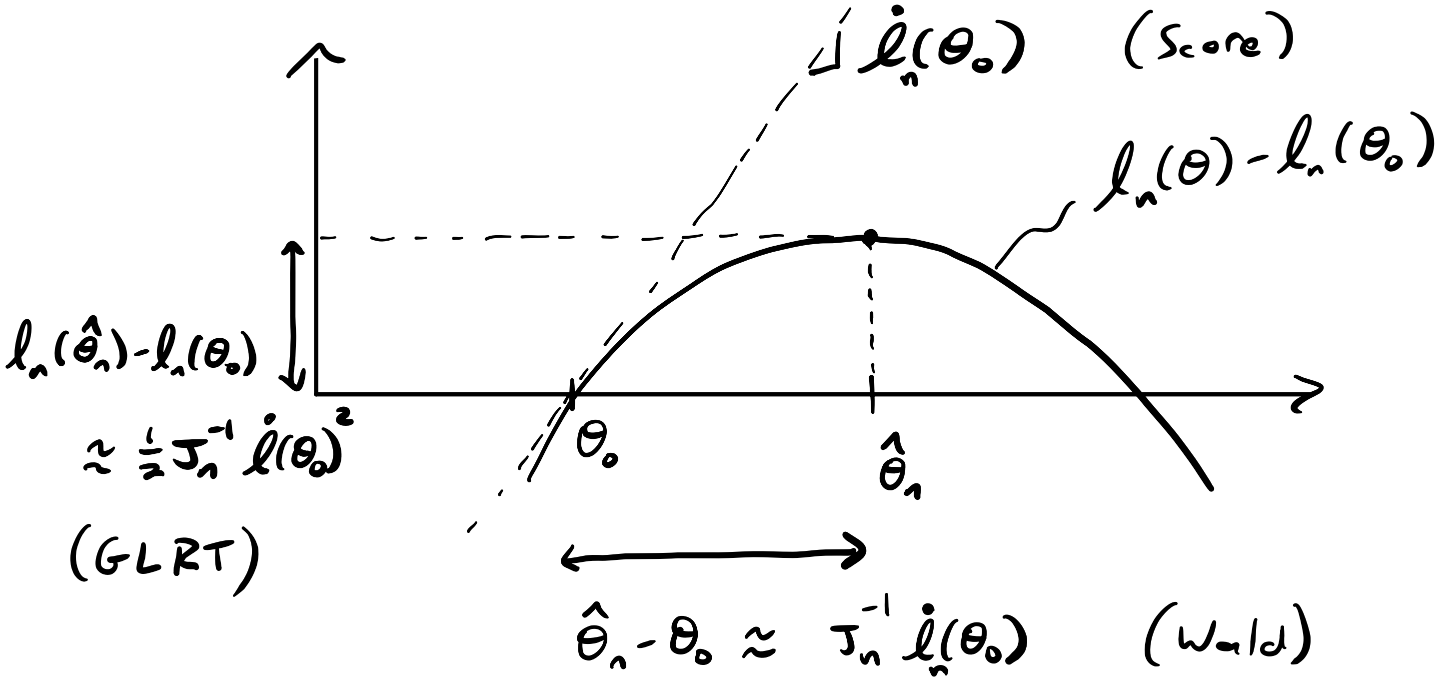
$$= n \|\mathrm{Proj}_{\Theta}^{\perp 0}(\hat{\theta}_n)\|^2$$

$$\Rightarrow \chi^2_{d-d_0}$$

Asymptotic Equivalence

Recall quadratic approx. picture $(d=1)$:

$$\ell_n(\theta)-\ell_n(\theta_0) \approx \dot{\ell}_n(\theta_0)(\theta-\theta_0) + \frac{1}{2} J_n(\theta_0)(\theta-\theta_0)^2$$



For large n,

$$\underset{\text{(GLRT)}}{\ell_n(\hat{\theta}_n)-\ell_n(\theta_0)} \approx \|J_n(\theta_0)^{1/2}(\hat{\theta}_n-\theta_0)\|^2$$

$$\approx \underset{\text{(wald)}}{\|\hat{J}_n^{1/2}(\hat{\theta}_n-\theta_0)\|^2} \qquad \approx \underset{\text{(Score)}}{\|J_n(\theta_0)^{-1/2}\nabla \ell_n(\theta_0)\|^2}$$

Asymptotic Relative Efficiency (ARE)

Suppose $\hat{\theta}_n^{(i)}$ $i=1,2$ are two asy. Normal estimators of $\theta_0 \in \mathbb{R}$, with

$$\sqrt{n}\left(\hat{\theta}_n^{(i)} - \theta_0\right) \Rightarrow N(0, \sigma_i^2)$$

The ARE of $\hat{\theta}^{(2)}$ wrt $\hat{\theta}^{(1)}$ is σ_1^2 / σ_2^2

e.g. if $\sigma_2^2 = 2\sigma_1^2$ then $\hat{\theta}^{(2)}$ is 50% as efficient

Interpretation: Suppose $\sigma_1^2 / \sigma_2^2 = \gamma \in (0,1)$

Then for large n,

$$\hat{\theta}^{(1)}_{\lfloor \gamma n \rfloor}(X_1, \dots, X_{\lfloor \gamma n \rfloor}) \overset{D}{\approx} \hat{\theta}_n^{(2)}(X_1, \dots, X_n) \approx N\left(\theta, \frac{\sigma_2^2}{n}\right)$$

Using $\hat{\theta}^{(2)}$ is like throwing away $100(1-\gamma)\%$ of the data and then using $\hat{\theta}^{(1)}$