▨ Various distributions

(a) Discrete distributions

① Bernoulli / Binomial distributions

- $X \sim \text{Bernoulli}(p)$ $(0 \leq p \leq 1)$ if

$$\mathbb{P}(X=1) = p \quad \text{and} \quad \mathbb{P}(X=0) = 1-p.$$

Example) Coin tossing (possibly unfair)

* $\mathbb{E}X = p$, $\text{Var}X = p(1-p)$

- $X \sim \text{Binomial}(n,p)$ (or $B(n,p)$) $(n \in \mathbb{N}, 0 \leq p \leq 1)$ if

$$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \cdots, n$$

* $\mathbb{E}X = np$, $\text{Var}X = np(1-p)$

* If $X_1, \cdots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$,

$$X_1 + \cdots + X_n \sim \text{Binomial}(n,p)$$

② Geometric distributions

- $X \sim \text{Geometric}(p)$ (or $\text{Geom}(p)$) if

$$\mathbb{P}(X=k) = p(1-p)^{k-1}, \quad k = 1, 2, \cdots$$

Example) Coin tossing (possibly unfair)

↳ the number of tossing until we get a "heads"

proof ⌐ $* \ \mathbb{E}X = \frac{1}{p}$, $\mathrm{Var}\,X = \frac{1-p}{p^2}$

$$\mathbb{P}(X=k) = \exp\{(k-1)\log(1-p) + \log p\} \times 1$$

natural parameter $\eta = \log(1-p)$ $\rightsquigarrow$ $p = 1-e^{\eta}$

sufficient statistic $T(X) = X-1$

base density $h(k) = 1$

cumulant-generating function

$$A(\eta) = -\log p = -\log(1-e^{\eta})$$

$$\mathbb{E}_\eta T(X) = A'(\eta) = \frac{e^{\eta}}{1-e^{\eta}}$$

$$\mathbb{E}_\eta X = \mathbb{E}_\eta T(X) + 1 = \frac{1}{1-e^{\eta}} = \frac{1}{p}$$

$$\mathrm{Var}_\eta T(X) = A''(\eta) = \frac{e^{\eta}(1-e^{\eta}) + e^{2\eta}}{(1-e^{\eta})^2} = \frac{e^{\eta}}{1-e^{\eta}}$$

$$\mathrm{Var}_\eta X = \mathrm{Var}_\eta T(X) = \frac{1-p}{p^2}$$

③ Poisson distributions

$X \sim \mathrm{Poisson}(\lambda)$ $(\lambda > 0)$ if

$$\mathbb{P}(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \cdots$$

= The limit of $B(n,p)$ in the sense that

$n \to \infty,\ p \to 0,\ np \to \lambda$

Suppose, $X \sim B(n,p)$

$$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \times p^k (1-p)^{n-k}$$

$$= \frac{1}{k!}\, 1 \times \left(1-\frac{1}{n}\right) \cdots \left(1-\frac{k-1}{n}\right) (np)^k \left(1-\frac{np}{n}\right)^{n-k}$$

$$\xrightarrow[\substack{n\to\infty \\ p\to 0 \\ np\to\lambda}]{} \frac{1}{k!} \lambda^k e^{-\lambda}$$

* $\mathbb{E}X = \lambda$, $\mathrm{Var}\,X = \lambda$

* If $X_1 \sim \mathrm{Poisson}(\lambda_1)$, $X_2 \sim \mathrm{Poisson}(\lambda_2)$,

and they are independent,

$X_1 + X_2 \sim \mathrm{Poisson}(\lambda_1 + \lambda_2)$.

(iii) Continuous distributions

① Exponential distributions

$X \sim \mathrm{Exp}(\lambda)$ $(\lambda > 0)$ if

scale

the pdf of X is

$$f(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}, \quad x > 0$$

[cf. Some people use the following notation]

$X \sim \mathrm{Exp}(\underbrace{\lambda}_{\text{rate}}) \quad (\lambda > 0)$ if

the pdf of X is

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

$*$ $\mathbb{E}X = \lambda$, $\mathrm{Var}X = \lambda^2$

proof

$$f(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} = \exp\left(x \times \left(-\frac{1}{\lambda}\right) + \log\left(\frac{1}{\lambda}\right)\right) \times 1$$

natural parameter $\eta = -\frac{1}{\lambda}$

sufficient statistic $T(X) = X$

base density $h(x) = 1$

cumulant-generating function

$$A(\eta) = -\log\left(\frac{1}{\lambda}\right) = -\log(-\eta)$$

$$\mathbb{E}_\eta X = \underline{\mathbb{E}_\eta T(X) = A'(\eta)} = -\frac{1}{\eta} = \lambda$$

$$\mathrm{Var}_\eta X = \underline{\mathrm{Var}_\eta T(X) = A''(\eta)} = \frac{1}{\eta^2} = \lambda^2$$

② Gamma distributions

$X \sim \mathrm{Gamma}(k, \theta) \quad (\overbrace{k}^{\text{shape}}, \overbrace{\theta}^{\text{scale}} > 0)$

the pdf of X is

$$f(x) = \frac{1}{\underbrace{\Gamma(k)}_{\text{Gamma function}}\theta^k} x^{k-1} e^{-\frac{x}{\theta}}, \quad x > 0$$

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x}\,dx \quad , k>0$$

$* \ \Gamma(k+1) = k\Gamma(k) \ , \ k>0$

$* \ \mathbb{E}X = ?$, $\mathrm{Var}X = ?$

$*$ If $X_1 \sim \mathrm{Gamma}(k_1, \theta)$, $X_2 \sim \mathrm{Gamma}(k_2, \theta)$,

and they are independent,

$X_1 + X_2 \sim ?$

$* \ \mathrm{Gamma}(1,\theta) \equiv \mathrm{Exp}(\theta)$

③ Beta distributions

$X \sim \mathrm{Beta}(\alpha, \beta)$ if

the pdf of X is

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in (0,1)$$

$$* \ \mathbb{E}X = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

proof

$$1 = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$\Rightarrow \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\mathbb{E}X = \int_0^1 x \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \int_0^1 x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}$$

$$= \frac{\alpha}{\alpha+\beta}$$

$$\mathbb{E}X^2 = \int_0^1 x^2 \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\mathrm{Var}\,X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

* If $X_1 \sim \mathrm{Gamma}(\alpha_1, \beta)$, $X_2 \sim \mathrm{Gamma}(\alpha_2, \beta)$,

and they are independent.

$$\frac{X_1}{X_1+X_2} \sim \text{Beta}(\alpha_1, \alpha_2)$$

④ Normal distributions

$X \sim N(\mu, \sigma^2)$ $(\mu \in \mathbb{R}, \sigma > 0)$ if

the pdf of X is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad x \in \mathbb{R}$$

* $\mathbb{E}X = \mu$, $\mathrm{Var}X = \sigma^2$

⑤ Other important distributions

Multivariate normal distributions

t-distributions (or Student's t-distribution)

Chi-squared distributions

F distributions

⋮

We may have a chance to cover them later.

(when we learn testing)