Outline

① Probability generating function (PGF)

② Moment generating function (MGF)

③ Change of variables for pdf

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① Probability generating function

X : a random variable that takes values in $\mathbb{Z}^+$

We define the probability generating function of X as

$$G_X(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} s^k \, \mathbb{P}(X=k) \quad \text{for } -1<s<1$$

Note that

In fact, converges absolutely

the sequence converges for $s \in (-1,1)$. → well-defined

Example) $X \sim \text{Binomial}(n,p)$

$$G_X(s) = \sum_{k=0}^{n} s^k \times \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (sp)^k (1-p)^{n-k} = (1-p+sp)^n$$

Denote r by the rate of convergence of $G_X(s)$.

$(r \geq 1)$

$$G_X(s) = \sum_{k=0}^{\infty} \mathbb{P}(X=k)\, s^k \quad \text{for } -r<s<r.$$

$$G_X'(s) = \sum_{k=0}^{\infty} k\,\mathbb{P}(X=k)\, s^{k-1} \quad \text{for } -r<s<r$$

$$G_X''(s) = \sum_{k=1}^{\infty} k(k-1) \times \mathbb{P}(X=k)\, s^{k-2} \quad \text{for } -r<s<r$$

$$\vdots$$

$$G_X^{(m)}(s) = \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1)\,\mathbb{P}(X=k)\, s^{k-m}$$

$$\text{for } -r<s<r.$$

Put $s \leftarrow 0$.

$$G_X^{(m)}(0) = m(m-1) \times \cdots \times 1 \cdot \mathbb{P}(X=m)$$

$$\Rightarrow \mathbb{P}(X=m) = \frac{1}{m!} G_X^{(m)}(0).$$

Put $s \leftarrow 1$.

$$G_X^{(m)}(1) = \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1)\,\mathbb{P}(X=k)$$

$$= \mathbb{E}[X(X-1)\cdots(X-m+1)]$$

Example) $X \sim \text{Binomial}(n,p)$

$$G_X(s) = (1-p+sp)^n \quad \text{for } s \in \mathbb{R}$$

$$\mathbb{E}[X(X-1)\cdots(X-m+1)] = G_X^{(m)}(1)$$
$$= n(n-1)\cdots(n-m+1)p^m(1-p+p)^n$$
$$= n(n-1)\cdots(n-m+1)p^m$$

② Moment generating function

X : a random variable

Suppose $\mathbb{E}[e^{uX}] < +\infty$ for $-r<u<r$.

We define the moment generating function of X as

$$M_X(u) = \mathbb{E}[e^{uX}] \quad \text{for } -r<u<r.$$

Example) $X \sim N(\mu, \sigma^2)$

Recall that the pdf of X is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad x\in\mathbb{R}$$

$$\Rightarrow M_X(u) = \mathbb{E}[e^{uX}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} + ux\right\} dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu-u\sigma^2)^2 + u\mu + \frac{u^2}{2}\sigma^2\right\} dx$$
$$= \exp\left\{u\mu + \frac{u^2}{2}\sigma^2\right\}.$$

- For every $m \in \mathbb{N}$,

$$\mathbb{E}[X^m] = M_X^{(m)}(0)$$

called "moment"

- $M_X(u) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbb{E}[X^k] u^k$ for $-r < u < r$

- X, Y : random variables

If $\exists \varepsilon > 0$ s.t. $M_X(u) = M_Y(u)$ for $-\varepsilon < u < \varepsilon$,

then X and Y have the same distribution.

Example) $X \sim N(0, \sigma^2)$

$$M_X(u) = \exp\left\{ \frac{u^2}{2} \sigma^2 \right\} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{u^2}{2} \sigma^2 \right)^k$$

$$\left(\exp(t) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \right)$$

$$= \sum_{k=0}^{\infty} \frac{\sigma^{2k}}{2^k \cdot k!} \times u^{2k} \quad \text{for } u \in \mathbb{R}.$$

$$\Rightarrow \mathbb{E}[X^k] = \begin{cases} 0 & \text{if } k : \text{odd} \\ \dfrac{k!}{2^{\frac{k}{2}} \left(\frac{k}{2}\right)!} \times \sigma^k & \text{if } k : \text{even} \end{cases}$$

③ Change of variables for pdf

X : a random variable, $Y = u(X)$

(i) X : discrete

$$P_Y(y) = \sum_{x : u(x) = y} P_X(x)$$

(ii) X : continuous

determinant

$$P_Y(y) = \sum_{x : u(x) = y} P_X(x) \left| \frac{dy}{dx} \right|^{-1}$$

not rigorous

$$\left(\begin{aligned} &\mathbb{P}(Y \in (y-dy, y+dy)) = \mathbb{P}(u(X) \in (y-dy, y+dy)) \\ &\quad = \sum_{x : u(x) = y} \mathbb{P}(X \in (x-dx, x+dx)) \\ &\Rightarrow P_Y(y)\,|dy| = \sum_{x : u(x) = y} P_X(x)\,|dx| \end{aligned} \right)$$

independent

Example) ① $X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$

$$Y = X_1 + X_2$$

$$P_Y(y) = \sum_{x_1 + x_2 = y} \frac{e^{-\lambda_1} \lambda_1^{x_1}}{x_1!} \times \frac{e^{-\lambda_2} \lambda_2^{x_2}}{x_2!}$$

$$= \frac{e^{-\lambda_1 - \lambda_2}}{y!} \sum_{x_1 = 0}^{y} \binom{y}{x_1} \lambda_1^{x_1} \lambda_2^{y - x_1}$$

$$= \frac{e^{-\lambda_1 - \lambda_2} (\lambda_1 + \lambda_2)^y}{y!} \quad \text{for } y = 0, 1, 2, \cdots$$

$$\Rightarrow Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

③ $X \sim N(0,1)$, $Y = X^2$

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$p_Y(y) = \sum_{x:x^2=y} p_X(x) \left| \frac{dy}{dx} \right|^{-1}$$

$$= \sum_{x:x^2=y} p_X(x) \times \frac{1}{2|x|}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \times \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \quad \text{for } y > 0$$

$$\left(\begin{array}{l} X \sim \text{Gamma}(k, \theta) \quad (k, \theta > 0) \\ \text{the pdf of } X \text{ is } f(x) = \dfrac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}} \; , \; x > 0 \end{array} \right)$$

$$\Rightarrow Y \sim \text{Gamma}\left(\frac{1}{2}, 2\right)$$

$$\left(\chi^2(1) \equiv \text{Gamma}\left(\frac{1}{2}, 2\right) \right)$$

④ Order statistics

Assume $X_1, \cdots, X_n \overset{iid}{\sim}$ some continuous distribution with density f

(We can ignore the case $X_i = X_j$)

Let $Y = (X_{(1)}, \cdots, X_{(n)})$

Then,

$$p_Y(y_1, \cdots, y_n) = n!\, f(y_1) \cdots f(y_n)\, \mathbb{1}(y_1 < \cdots < y_n)$$

(sketch of proof)

$$\begin{cases} \mathcal{X} = \{(x_1, \cdots, x_n) : f(x_i) > 0,\ i = 1, \cdots, n\} \\ u(x_1, \cdots, x_n) = (x_{(1)}, \cdots, x_{(n)}) \end{cases}$$

For each $\pi \in S_n$ ~ permutation of size n

let $\mathcal{X}^\pi = \{(x_1, \cdots, x_n) : f(x_i) > 0,\ i = 1, \cdots, n$ and $x_{\pi(1)} < \cdots < x_{\pi(n)}\}$.

Note that $\mathcal{X} = \biguplus_{\pi \in S_n} \mathcal{X}^\pi$.

Also,

$$u|_{\mathcal{X}^\pi}(x_1, \cdots, x_n) = (x_{\pi(1)} < \cdots < x_{\pi(n)})$$

$$\overset{\text{let}}{:=} u^\pi(x_1, \cdots, x_n).$$

Thus, for $y_1 < \cdots < y_n$,

$$p_Y(y_1, \cdots, y_n) = \sum_\pi p_X(u^{\pi^{-1}}(y_1, \cdots, y_n)) \left| \frac{dy}{dx} \right|^{-1}$$

permutation matrix

det =

$$= \sum_\pi f(y_{\pi^{-1}(1)}) \cdots f(y_{\pi^{-1}(n)})$$

$$= \sum_\pi f(y_1) \cdots f(y_n)$$

$$= n!\, f(y_1) \cdots f(y_n) \quad \blacksquare$$

Let F be the CDF.

- $$P_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1} f(x)[1-F(x)]^{n-r}$$

  for $1 \le r \le n$

- $$P_{X_{(r)},X_{(s)}}(x,y) = \frac{n!}{(r-1)!(s-1-r)!(n-s)!}$$

  $$\times [F(x)]^{r-1} f(x)[F(y)-F(x)]^{s-1-r} f(y)[1-F(y)]^{n-s}$$

  for $1 \le r < s \le n$ and $x < y$.

(sketch of proof)

$$P_{X_{(r)}}(x) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} n!\, f(y_1)\cdots f(y_{r-1}) f(x) f(y_{r+1})\cdots f(y_n)$$

$$\times \underbrace{1(y_1<\cdots<y_{r-1}<x<y_{r+1}<\cdots<y_n)}_{=\,1(y_1<\cdots<y_{r-1}<x)\times 1(x<y_{r+1}<\cdots<y_n)} dy_1\cdots dy_n$$

$$= n!\, f(x)\cdot \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \underbrace{f(y_1)\cdots f(y_{r-1})}_{\text{invariant w.r.t permutation of } y_1,\cdots,y_{r-1}} 1(y_1<\cdots<y_{r-1}<x)\, dy_1\cdots dy_{r-1}$$

$$\cdot \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \underbrace{f(y_{r+1})\cdots f(y_n)}_{\text{invariant w.r.t permutation of } y_{r+1},\cdots,y_n} 1(x<y_{r+1}<\cdots<y_n)\, dy_{r+1}\cdots dy_n$$

$$= n!\, f(x) \times \frac{1}{(r-1)!}\int_{-\infty}^{x}\cdots\int_{-\infty}^{x} f(y_1)\cdots f(y_{r-1})\, dy_1\cdots dy_{r-1}$$

$$\cdot \frac{1}{(n-r)!}\int_{x}^{\infty}\cdots\int_{x}^{\infty} f(y_{r+1})\cdots f(y_n)\, dy_{r+1}\cdots dy_n$$

$$= n!\, f(x) \times \frac{1}{(r-1)!} F(x)^{r-1} \cdot \frac{1}{(n-r)!} [1-F(x)]^{n-r}$$

$$= \frac{n!}{(r-1)!\,(n-r)!} [F(x)]^{r-1} f(x) [1-F(x)]^{n-r}$$